Partial Solution Set, Leon §6.3

6.3.1a We are to find a matrix X that diagonalizes $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues of A in this case are $\lambda_1 = 1$ and $\lambda_2 = -1$. By Theorem 6.3.2, such a matrix X exists, and the columns of X are eigenvectors of A corresponding to the eigenvalues. In the usual fashion, we find eigenvectors $\mathbf{x}_1 = (1,1)^T$ and $\mathbf{x}_2 = (1,-1)^T$. So $X = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, and $X^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. It is easy to verify that $X^{-1}AX = D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, so $A = XDX^{-1}$.

6.3.1b As in (a), but with $A = \begin{bmatrix} 5 & 6 \\ -2 & -2 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$. An eigenvector associated with λ_1 is $\mathbf{x}_1 = (-3,2)^T$, and an eigenvector associated with λ_2 is $\mathbf{x}_2 = (-2,1)^T$. So $X = \begin{bmatrix} -3 & -2 \\ 2 & 1 \end{bmatrix}$, and $X^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & 13 \end{bmatrix}$. As expected, $D = X^{-1}AX = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

6.3.1d As in (a) and (b), but now $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}$. The eigenvalues are $\lambda_1 = 2,\ \lambda_2 = 1$, and and $\lambda_3 = -1$. The associated eigenvectors are (any scalar multiples of) $\mathbf{x}_1 = (1,0,0)^T, \mathbf{x}_2 = (-2,1,0)^T$, and $\mathbf{x}_3 = (1,-3,3)^T$. So $X = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 3 \end{bmatrix}$, and $X^{-1} = \begin{bmatrix} 1 & 2 & \frac{5}{3} \\ 0 & 1 & 1 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$.

6.3.4a Given $A = \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix}$, find a matrix B such that $B^2 = A$. In other words, find a square root for A. This is easily done using diagonalization, i.e., once we factor $A = XDX^{-1}$, the matrix B that we are after is $B = XCX^{-1}$, where $C = (c_{ij}) = (\sqrt{d_{ij}})$. We start by finding the eigenvalues of A. Since A is clearly singular, we know that $\lambda_1 = 0$ is an eigenvalue; we find that the other is $\lambda_2 = 1$. At this point, we can either press on, finding a pair of eigenvectors, or observe that if D is diagonal with $d_{11} = 0,\ d_{22} = 1$, then either $C = -D$ or $C = D$. This suggests that both A and $-A$ might have the desired property, and this in fact turns out to be the case.

6.3.4b This is a bit more work, but at least $A = \begin{bmatrix} 9 & -5 & 3 \\ 0 & 4 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ is triangular and we don't have to work hard to find eigenvalues $9, 4$, and 1. The usual process leads to $X = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ and $X^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Thus $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 9 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Setting

$$B = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix},$$

it is easily verified that $B^2 = A$.

6.3.6 Let A be a diagonalizable matrix whose eigenvalues are all either 1 or -1. Show that $A^{-1} = A$.

Proof: There is no question about the existence of A^{-1}, since A has strictly nonzero eigenvalues. Since A is diagonalizable, we can construct A such that $A = XDX^{-1}$, where D is a diagonal matrix containing the eigenvalues of A, i.e., every diagonal entry is either 1 or -1. But then $D^{-1} = D$, so

$$A^{-1} = \left(XDX^{-1}\right)^{-1} = XD^{-1}X^{-1} = XDX^{-1} = A,$$

which is what we needed to show. □

6.1.7 Show that any 3×3 matrix of the form $\begin{bmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & b \end{bmatrix}$ is defective.

Proof: By inspection, the eigenvalues of A are a, with algebraic multiplicity two, and b. We must show that the geometric multiplicity of a is less than two. To do this, we first form $A - aI = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & b-a \end{bmatrix}$. Notice that if $\mathbf{x} = (x_1, x_2, x_3)^T$ is any eigenvector for a, then $x_2 = x_3 = 0$, so $\mathbf{x} = s\mathbf{e}_1$ for some scalar s. It follows that the geometric multiplicity of a is $\dim(N(A - aI)) = 1 < 2$, and A is defective. □

Note: It would suffice to observe that $\operatorname{rank}(A - aI) = 2$, which forces $\operatorname{null}(A - aI) = 1$.

6.3.8c We are given a square matrix

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 2 & -1 & \alpha \end{bmatrix},$$

and challenged to find values of α for which A is defective. We begin by calculating the characteristic equation of A, which is

$$P(\lambda) = (\alpha - \lambda)\left[(1-\lambda)^2 - 4\right] = (\alpha - \lambda)(\lambda - 3)(\lambda - 1).$$

The eigenvalues are $3, -1$, and α. If $\alpha \notin \{-1, 3\}$, then A is diagonalizable. We must check the dimension of $N(A - \alpha I)$ for $\alpha = -1$ and $\alpha = 3$. If $\alpha = 3$, then

$$A - \alpha I = \begin{bmatrix} -2 & 2 & 0 \\ 2 & -2 & 0 \\ 2 & -1 & 0 \end{bmatrix},$$

a matrix of rank 2. It follows that $\dim(N(A - 3I)) = 1$, but then A must be defective. The procedure for $\alpha = -1$ follows the same lines, and yields a similar conclusion.

6.3.9 Let A be a 4×4 matrix, and let λ be an eigenvalue of multiplicity 3. If $A - \lambda I$ has rank 1, is A defective?

Solution:

No, if $A - \lambda I$ has rank 1, then the geometric multiplicity of λ is $4 - 1 = 3$, equal to the algebraic multiplicity of λ.

6.3.17 Let A be a diagonalizable $n \times n$ matrix. Prove that if B is any matrix that is similar to A, then B is diagonalizable.

Proof: Suppose A and B are as described, and let X be a matrix that diagonalizes A. So $A = XDX^{-1}$ and $B = S^{-1}AS$ for some nonsingular S. But then

$$B = S^{-1}XDX^{-1}S = (S^{-1}X)D(S^{-1}X)^{-1}$$

is diagonalizable. $\square$

As a corollary, we get another proof that similar matrices have the same eigenvalues, along with a strong suggestion that in general they have different eigenvectors.